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Optimum life test plan for products sold under warranty having Type-I generalized hybrid censored Weibull distributed lifetimes

Jimut Bahan Chakrabarty[1]
Shovan Chowdhury [2]
Soumya Roy [3]

[1] Doctoral Scholar, Quantitative Methods and Operations Management at the Indian Institute of Management Kozhikode, Kozhikode, India. IIMK Campus P.O., Kozhikode, Kerala 673570, India; Email: jimutb08fpm@iimk.ac.in.

[2] Associate Professor, Quantitative Methods and Operations Management at the Indian Institute of Management Kozhikode,Kozhikode, India. IIMK Campus P.O., Kozhikode, Kerala 673570, India; Email:shovanc@iimk.ac.in PhoneNumber(+91)495-2809119

[3] Assistant Professor, Quantitative Methods and Operations Management at the Indian Institute of Management Kozhikode,Kozhikode, India. IIMK Campus P.O., Kozhikode, Kerala 673570, India; Email:soumya@iimk.ac.in PhoneNumber(+91)495-2809109

Abstract: In order to ensure maintenance of a certain quality level for a product, choosing a suitable life test plan is immensely essential. Since life testing includes as well as impacts various costs, it is important to design a life testing plan incorporating the relevant costs. In this paper, a model is proposed to obtain an optimal life testing plan for non-repairable products sold under general rebate warranty. The proposed model determines the optimal plan by minimizing the suitable costs involved. Type-I generalized hybrid censoring setup for products having Weibull distributed lifetimes is considered for the model presented. Considering both producer's and consumer's risk, a constrained optimization approach is followed and appropriate analysis techniques are employed in obtaining the optimal solution. An extensive simulation study is performed for numerical illustration. In order to analyze the sensitivity of the optimal solution due to mis-specification of parameter values and cost components, a well designed sensitivity analysis is incorporated using parameter estimates from real life hybrid censored data set.

Keywords: Life testing plan, Weibull distribution, Type-I generalized hybrid censoring, General rebate warranty, Constrained optimization.

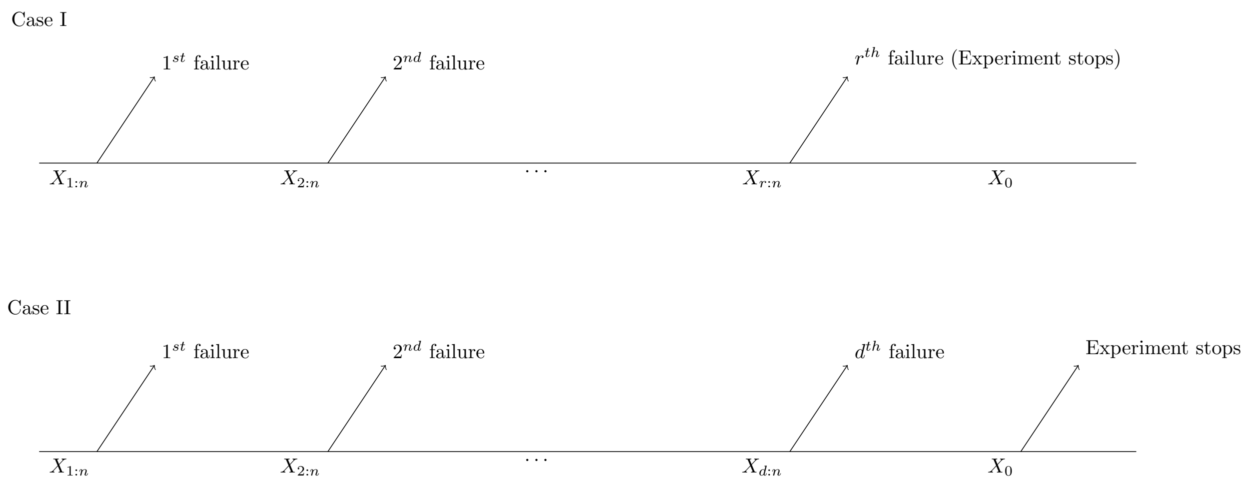
1 Introduction

Acceptance sampling as an approach for industrial quality control has widespread usage across manufacturing industries. The selection of a lot or batch of raw materials or any other component units is usually decided using this technique. Wu *et al* (2015) mentioned it as a technique that shrinks the gap between expected and the actual quality of manufactured goods. The quality of a product has multifaceted importance for a business. Quality helps in strengthening customers' trust for a product and thereby impact sales which explains the importance of acceptance sampling plans and its extensive usage in manufacturing industry. Acceptance sampling plan can be simplistically described as follows: Consider fresh arrival of a shipment of raw materials at a manufacturing unit. To test the pre-specified quality characteristics, a sample is drawn from the shipment. From the information derived upon testing the sample, a decision is reached on whether to accept or reject the lot.

For consumer durable products, lifetime is one of the indispensable quality attribute. It should be kept in mind that lifetime as a quality attribute is not an instantaneously acquired dimensional measurement. Censoring is usually applied to collect lifetime data (Wu and Huang, 2017). Since censoring techniques are employed while testing lifetime, the response values are not observable for all the units under study. Among all the censoring schemes, Type-I and Type-II censoring schemes are most commonly discussed in life testing literature. The two censoring schemes differ in the criterion through which the experiment is terminated. Type-I censoring scheme is terminated at a pre-decided time X_0, hence it is also called time censoring. On the other hand Type-II censoring, which is popularly known as failure censoring, is terminated after a pre-chosen number of failures (r) are observed. Combining the two censoring schemes, Epstein (1954) came up with hybrid censoring which later came to be known as Type-I hybrid censoring. In this censoring scheme, if n identical units are put on test having ordered lifetimes $X_{1:n}, ..., X_{n:n}$ respectively, then the experiment is aborted either when a pre-chosen number $r < n$ out of n items has failed or when a pre-determined time X_0 has elapsed. Therefore, the experiment can be terminated at a random time $X^* = min\{X_{r:n}, X_0\}$. It must be noted that when $r = n$, Type-I hybrid censoring transforms to Type-I censoring scheme whereas when $X_0 \to \infty$, it transforms to Type-II censoring scheme. One of the following two types of observations can be witnessed under Type-I hybrid censoring scheme.

Case I: $\{X_{1:n} < ... < X_{r:n}\}$ if $X_{r:n} < X_0$.
Case II: $\{X_{1:n} < ... < X_{d:n} < X_0\}$ if $d + 1 \leq r < n$ and $X_0 \leq X_{r:n}$.

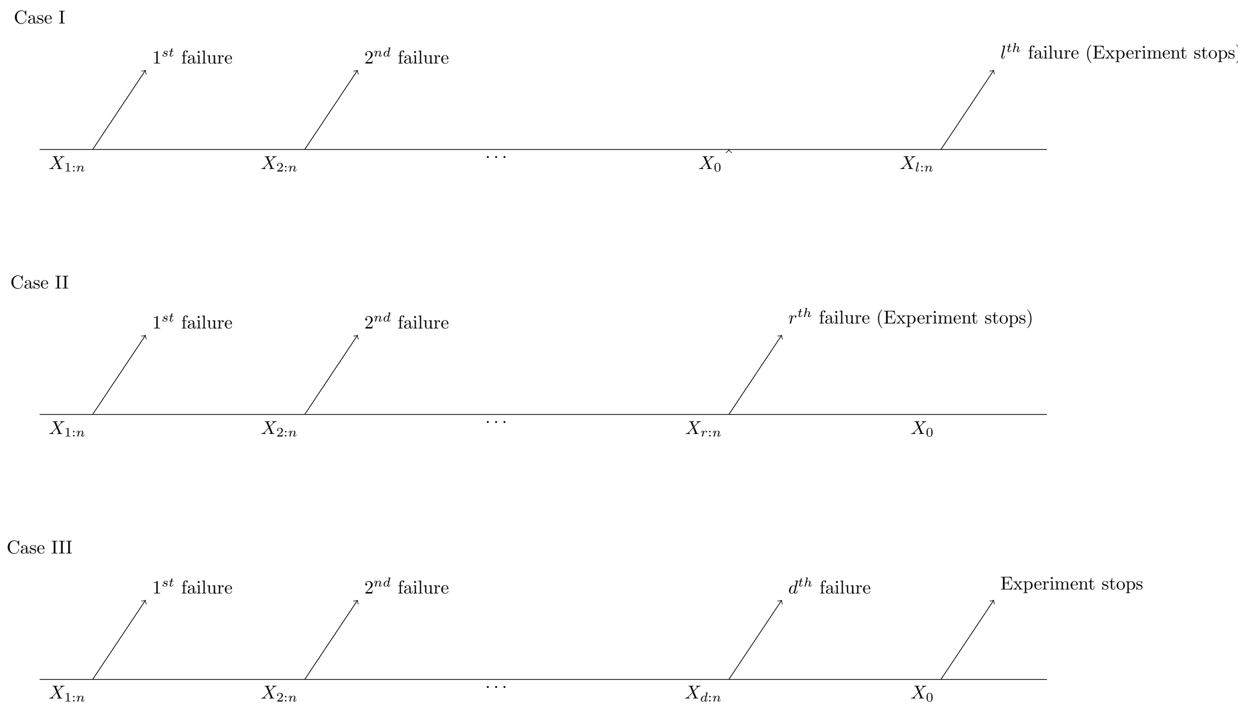
Figure 1: Schematic illustration of Type-I hybrid censoring scheme.



The problem that arises out of Type-I hybrid censoring is that, there is a high chance of observing very few or no failures if the mean lifetime of experimental units is greater than the censoring time. This leads to the introduction of Type-II hybrid censoring scheme by Childs *et al* (2003) which considered $X^* = max\{X_{r:n}, X_0\}$ as the termination time. The test completion may take a very long time in Type-II hybrid censoring scheme. To overcome the limitations of both Type-I and Type-II hybrid censoring schemes, Chandrasekar *et al.* (2004) has come up with Type-I and Type-II generalized hybrid censoring schemes. Under Type-I generalized hybrid censoring scheme (GHCS), the experiment is terminated at $X^* = min\{X_{r:n}, X_0\}$ only if l failures are observed before time X_0. If l failures occur after time X_0 has elapsed, then the experiment is terminated at $X_{l:n}$. One of the following three types of observations can be witnessed under Type-I generalized hybrid censoring scheme.

Case I: $\{X_{1:n} < ... < X_{l:n}\}$ if $X_{l:n} > X_0$.
Case II:$\{X_{1:n} < ... < X_{r:n}\}$ if $X_{r:n} < X_0$.
Case III: $\{X_{1:n} < ... < X_{d:n} < X_0\}$ if $l \leq d$, $d + 1 \leq r < n$ and $X_0 \leq X_{r:n}$.

Figure 2: Schematic illustration of Type-I generalized hybrid censoring scheme.



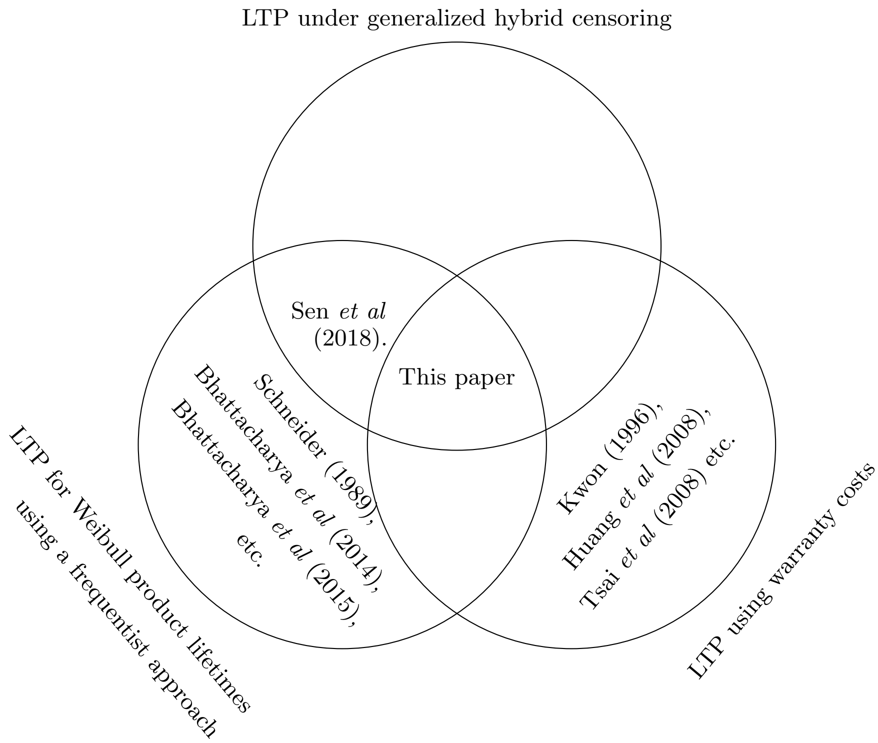
Several costs are associated with or impacted by a life test plan, hence minimizing the average aggregate costs involved is one of the most viable choices for a decision maker. Although the literature has paid enough attention towards development of life testing plans through a combination of different methods and censoring schemes, the appropriate choices of costs from a business point of view has seldom being dealt with. Gupta (1962) developed life test sampling plan for normal and lognormal distribution where experiment time is fixed in advance. Schneider (1989) under the assumption of lognormal and Weibull lifetimes has developed failure censored life test plans. Balasooriya and Balakrishnan (2000) has presented reliability sampling plan for the lognormal distribution based on progressively censored samples. Battacharya *et al* (2015) has minimized the asymptotic variance considering both producer's and consumer's risks under Weibull lifetimes. Dube *et al* (2011) has estimated the parameters of hybrid censored lognormal distribution using various estimation approaches and has proposed a life testing plan considering a pre-decided fixed total cost. Lin *et al* (2008) has employed a quadratic loss function under Bayesian setup to develop life testing plans for Type-I and Type-II hybrid censored samples. More recently, Sen *et al* (2018) under generalized hybrid censoring setup has used asymptotic variance minimization approach to determine optimal life test plan. Sen *et al* (2018) has proposed two solution approaches for Type-I generalized hybrid censoring, neither of them takes costs due to acceptance or rejection into consideration.

While designing a life testing plan from a cost perspective it is necessary to choose appropriate costs for arriving at a sampling plan which suffices the purpose. For a consumer durable product with warranty, it is rationale to have warranty cost in the design scheme. Employing a Bayesian approach Kwon (1996) is the first paper to include warranty cost as lot acceptance cost under Type-II censoring setup arguing the impact of acceptance of a lot on its future warranty. Kwon (1996) has chosen general rebate warranty to determine the acceptance cost. General rebate warranty is a warranty policy which combines the two most elementary warranty policies widely used for non-repairable products, free replacement warranty and pro-rata warranty. In free replacement warranty, the warranty services can be availed for free during the warranty period; whereas, in case of pro-rata warranty a proportional warranty fee is charged on a pro-rata basis. Later, Huang *et al* (2008), Tsai *et al* (2008), and Hsieh and Lu (2013) has adopted the idea of inclusion of warranty cost as acceptance cost under Type-II censoring setup. Hsieh and Lu (2013) has argued that for products sold under warranty, the extent of failure during warranty period influences production decisions and affects the acceptance cost for life testing significantly. Although warranty cost in life testing under Type-II censoring scheme has received some attention in the literature, but to the best of our knowledge warranty cost has not been included for designing life testing plans in any other censoring setup.

Frequent use of general rebate warranty and hybrid censoring scheme can be witnessed in automobile industry which gives practical viability to the inclusion of warranty costs while designing a life test plan. General rebate warranty can be seen widely in use for automobile products such as tyres, car batteries etc. Blachre *et al* (2015) has mentioned the use of different censoring schemes including hybrid censoring scheme in life testing of mechanical bearings. The paper has also mentioned that even though internal calculation tool based on physical models are used in manufacturing units, life testing using various censoring schemes is carried out to validate those models and obtain insights on the performance of the lot. The real life hybrid censored data set from Lawless (2005) used in this paper for the purpose of numerical illustration, also confirms the use of hybrid censoring in automobile industry. This has served as a practical motivation to design an appropriate life testing plan (LTP) combining generalized hybrid censoring scheme and general rebate warranty policy. It is to be noted that generalized hybrid censoring is a more generic censoring scheme which includes hybrid censoring as a special case and also overcomes the limitations of hybrid censoring. While warranty cost is taken as the lot acceptance cost for this study, the aggregate cost considered also comprises of three other costs which emerges from the literature, namely: rejection cost, time consumption cost, and inspection cost. The cost components are discussed in more detail in Section 3.

In this paper, we determine optimum life test plans (LTP) in presence of Type-I generalized hybrid censoring using a cost function approach for products sold under general rebate warranty scheme having Weibull lifetimes. The gap in the literature that we are trying to address is summarized in Figure 3. The approach followed in this paper in determining a life test plan accounts for the producer's and consumer's risk through a constrained optimization. The rest of the paper is as follows: In Section 2, we discuss the framework in detail. The relevant costs required to formulate the cost minimization problem is discussed in Section 3. In Section 4 we discuss the approach followed to obtain an optimum solution. An extensive simulation study conducted for the paper is discussed in Section 5. In Section 6 we discuss the sensitivity analysis conducted for the paper using Lawless (2003) data set pertaining to locomotive controls. We put down our conclusion in Section 7.

Figure 3: Diagrammatic representation of the surveyed literature.



2 Model

2.1 Weibull distributed lifetime

If the lifetime X of a testing unit follows Weibull distribution with probability density function (pdf), $f_X(x)$ given by

$$f_X(x) = k\lambda^k x^{k-1} e^{-(\lambda k)^k}; x > 0, \tag{2.1}$$

where $k > 0$ and $\lambda > 0$ are the respective shape and scale parameters. The corresponding cumulative distribution function (CDF), $F_X(x)$ can be written as

$$F_X(x) = 1 - e^{-(\lambda x)^k}; x > 0. \tag{2.2}$$

If we consider the transformation $T = \ln X$, the corresponding CDF of the of the extreme value distribution of T is given by

$$F_T(t) = 1 - e^{-e^{\frac{t-\mu}{\sigma}}}; -\infty < t < \infty, \tag{2.3}$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are the respective location and scale parameters given by $\mu = -\ln\lambda$ and $\sigma = \frac{1}{k}$. Let $X_1, X_2, ..., X_n$ be the lifetimes of n units to be put on test which follow Weibull distribution given by (2.2). Hence $T_1, T_2, ..., T_n$ will be the corresponding log-lifetimes which follow extreme value distribution given by (2.3). Suppose the ordered lifetimes of these n units be given by $T_{1:n} \leq T_{2:n} \leq ... \leq T_{n:n}$. If we consider Type-I generalized hybrid censoring framework, then the two random variables representing the number of failures and log-censoring time can be denoted by D and $\tau = min(T_{r:n}, T_0)$ respectively, where

$T_0 = \ln X_0$ and X_0 is the censoring time. Accordingly the data can be represented by $(T_{1:n}, T_{2:n}, ..., T_{D:n}, D)$. The likelihood function can be written as

$$L(\mu, \sigma) \propto \prod_{i=1}^{d} f_T(t_{i:n})(1 - F_T(\tau_0))^{n-d}, \tag{2.4}$$

where $t_{i:n}$, d, and τ_0 are the observed values of $T_{i:n}$, D, and τ respectively. Using results from Park and Balakrishnan (2009), the Fisher information matrix can be expressed as

$$\ell(\theta) = \ell_l(\theta) + \ell_{T_0 \wedge r}(\theta) - \ell_{T_0 \wedge l}(\theta)$$

where,

$$\ell_l(\theta) = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial\theta} \ln h_T(t)\right)' \left(\frac{\partial}{\partial\theta} \ln h_T(t)\right) \sum_{i=1}^{l} f_{i:n}(t) dt;$$

$$\ell_{T_0 \wedge r}(\theta) = \int_{-\infty}^{T_0} \left(\frac{\partial}{\partial\theta} \ln h_T(t)\right)' \left(\frac{\partial}{\partial\theta} \ln h_T(t)\right) \sum_{i=1}^{r} f_{i:n}(t) dt;$$

$$\ell_{T_0 \wedge l}(\theta) = \int_{-\infty}^{T_0} \left(\frac{\partial}{\partial\theta} \ln h_T(t)\right)' \left(\frac{\partial}{\partial\theta} \ln h_T(t)\right) \sum_{i=1}^{l} f_{i:n}(t) dt.$$

and $h_T(t) = \frac{1}{\sigma} e^{\frac{t-\mu}{\sigma}}$ and $f_{i:n}(t) = i\begin{pmatrix} n \\ i \end{pmatrix} \frac{1}{\sigma} e^{\frac{t-\mu}{\sigma} - (n-i+1)e^{\frac{t-\mu}{\sigma}}} \left(1 - e^{-e^{\frac{t-\mu}{\sigma}}}\right)^{i-1}$ are the hazard and density function of T and $T_{i:n}$ respectively. By the notation $a \wedge b$, the meaning conveyed is a or b whichever occurs earlier. On simplification, the Fisher information matrix becomes

$$\ell(\theta) = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial\theta} \ln h_T(t)\right)' \left(\frac{\partial}{\partial\theta} \ln h_T(t)\right) \sum_{i=1}^{l} f_{i:n}(t) dt + \int_{-\infty}^{T_0} \left(\frac{\partial}{\partial\theta} \ln h_T(t)\right)' \left(\frac{\partial}{\partial\theta} \ln h_T(t)\right) \sum_{i=l+1}^{r} f_{i:n}(t) dt.$$

On further simplification, the Fisher information matrix will be of the following form

$$\ell(\theta) = \begin{pmatrix} \ell_{11}(\theta) & \ell_{12}(\theta) \\ \ell_{21}(\theta) & \ell_{22}(\theta) \end{pmatrix}.$$

where,

$$\ell_{11}(\theta) = \frac{l}{\sigma^2} + \frac{1}{\sigma^2} \int_{-\infty}^{T_0} \sum_{i=l+1}^{r} f_{i:n}(t) dt,$$

$$\ell_{22}(\theta) = \frac{1}{\sigma^2} \left(\int_{-\infty}^{\infty} \left(1 + \frac{t-\mu}{\sigma}\right)^2 \sum_{i=1}^{l} f_{i:n}(t) dt + \int_{-\infty}^{T_0} \left(1 + \frac{t-\mu}{\sigma}\right)^2 \sum_{i=l+1}^{r} f_{i:n}(t) dt \right),$$

$$\ell_{12}(\theta) = \ell_{21}(\theta) = \frac{1}{\sigma^2} \left(\int_{-\infty}^{\infty} \left(1 + \frac{t-\mu}{\sigma}\right) \sum_{i=1}^{l} f_{i:n}(t) dt + \int_{-\infty}^{T_0} \left(1 + \frac{t-\mu}{\sigma}\right) \sum_{i=l+1}^{r} f_{i:n}(t) dt \right).$$

Hence the variance-covariance matrix which is found by inverting Fisher information matrix will be of the following form

$$\ell^{-1}(\theta) = \begin{pmatrix} \ell^{11}(\theta) & \ell^{12}(\theta) \\ \ell^{21}(\theta) & \ell^{22}(\theta) \end{pmatrix}.$$

2.2 Acceptance criterion

Lower specification limit (LSL) is of utmost importance when lifetime is considered as a quality attribute. By definition, lower specification limit is the lowest level of product quality that is within the acceptable range. If we consider lifetime as a quality attribute, then it is considered that higher the lifetime, better is its perceived quality and hence the upper specification limit becomes immaterial. But the product should at least last till a certain period of time so as to be in the acceptable quality range. This gives prominence to the lower specification limit when product lifetime is considered as a quality attribute.

Suppose the actual one-sided lower specification limit be L pertaining to items to be tested, then the items with lifetimes less than L should be considered nonconforming and hence unacceptable. The lifetime of the inspected product is assumed to be a Weibull distributed random variable X with unknown parameters. In lieu of actual lifetime of the product, log-lifetime ($T = \ln X$) is used, which leads to a smallest-extreme-value distribution when lifetime is Weibull with location parameter μ and scale parameter σ. As stated earlier, in generalized hybrid censoring scheme the experiment is conducted putting n units on test simultaneously and the experiment is terminated at $T^* = min\{X_{r:n}, T_0\}$ only if l failures are observed before time T_0. If l failures occur after time T_0 has elapsed, then the experiment is terminated at $X_{l:n}$. In practice, r is stated in terms of degree of censoring $q = 1 - \frac{r}{n}$. In a similar manner, l is expressed in terms of $q_1 = 1 - \frac{l}{r}$. Therefore, the fraction of nonconforming items, p, can be written as $p = Pr(T \leq L')$, where $L' = \ln L$. Using the lot acceptance criterion derived by Lieberman and Resnikoff (1955) we get the following expression

$$\hat{\mu} - k\hat{\sigma} > L'; \tag{2.5}$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the maximum likelihood estimates of μ and σ respectively and k is acceptability constant. The statistic $S = \hat{\mu} - k\hat{\sigma}$ is asymptotically normal with mean $E[S] = \mu - k\sigma$ and variance $Var[S] = \ell^{11}(\theta) + k^2\ell^{22}(\theta) - 2k\ell^{12}(\theta)$, where ℓ^{11}, ℓ^{22} and ℓ^{12} are elements of variance-covariance matrix and $\theta = (\mu, \sigma)$. So the standardized variate

$$U = \frac{\hat{\mu} - k\hat{\sigma} - (\mu - k\sigma)}{\sqrt{\ell^{11}(\theta) + k^2\ell^{22}(\theta) - 2k\ell^{12}(\theta)}} \tag{2.6}$$

is also asymptotically normal with mean 0 and variance 1. Therefore using arguments from Schneider (1989), the approximated OC curve can be represented by

$$\begin{aligned} \mathscr{L}(p) &= Pr(\hat{\mu} - k\hat{\sigma} > L' | p) \\ &= 1 - \Phi\left(\frac{\sigma(u_p + k)}{\sqrt{V}}\right); \end{aligned} \tag{2.7}$$

where, $V = \ell^{11}(\theta) + k^2\ell^{22}(\theta) - 2k\ell^{12}(\theta)$ and $u_p = \frac{L' - \mu}{\sigma}$ is the p^{th} quantile of the standard extreme value distribution corresponding to the nonconforming fraction $p = Pr((T - \mu)/\sigma \leq (L' - \mu)/\sigma)$ and $\mathscr{L}(p)$ is decreasing in p and Φ is standard normal distribution function.

If we consider α and β as producer's risk and consumer's risk respectively, then by fixing points $(p_\alpha, 1-\alpha)$ and (p_β, β) on the OC curve we can obtain the value of k and also n for any known value of T_0, where T_0 is the log of censoring time.

The expression for k thus obtained can be written as

$$k = \frac{u_{p_\alpha} z_{1-\beta} - u_{p_\beta} z_\alpha}{z_\alpha - z_{1-\beta}},$$

and the value of n can be found out by solving the following expression for known value of T_0

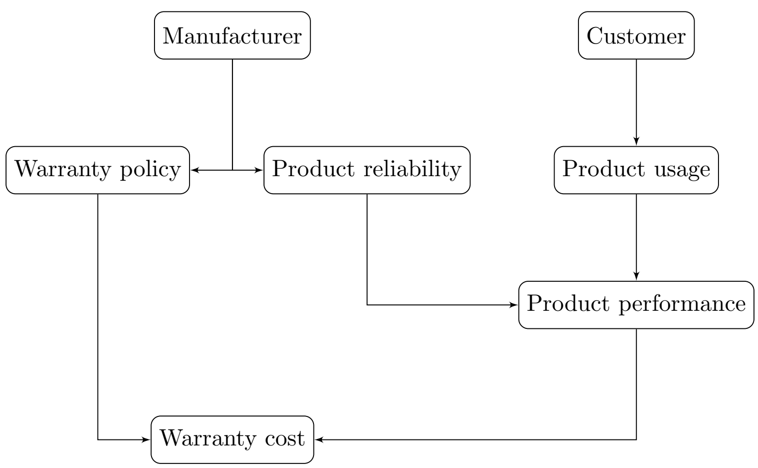
$$\frac{V}{\sigma^2}\left(\frac{z_\alpha - z_{1-\beta}}{u_{p_\alpha} - u_{p_\beta}}\right)^2 = 1, \tag{2.8}$$

where z_α and $z_{1-\beta}$ are α^{th} and $(1-\beta)^{th}$ quantiles of standard normal distribution and u_{p_α} and u_{p_β} are ${p_\alpha}^{th}$ and ${p_\beta}^{th}$ quantiles of the standard extreme value distribution corresponding to the nonconforming fractions p_α and p_β respectively.

3 Determining the cost function

Taking evidence from the literature (Kwon, 1995; Hsieh and Lu, 2013), we can arrive at four primary costs that has been consistently discussed while designing a life testing plan. The four costs as discussed above are as follows: $\langle a \rangle$ the cost of accepting a lot, $\langle b \rangle$ the cost of rejecting a lot, $\langle c \rangle$ the time-consumption cost, and $\langle d \rangle$ the inspection cost. Wu *et al* (2007) state that warranty cost is effected by product reliability. If we consider consumer durable products which are sold under general rebate warranty, the decision to accept a lot effects its warranty cost. Hence while designing a life test plan for such products, it is fruitful to keep in mind the costs related with the warranty policies (Kwon, 1995). The argument can further be substantiated by the following diagram used to characterize warranty cost by Murthy (2007).

Figure 4: Characterization of warranty cost.



Hence warranty cost can be substituted for the cost of acceptance of a lot (Kwon, 1996; Huang *et al*, 2008; Tsai *et al*, 2008; Hsieh and Lu, 2013). The warranty cost thus adopted as acceptance cost is a combination of two warranty policies, free-replacement warranty and pro-rata warranty. The combination of the two warranty policies is best known in the literature as general rebate warranty. The mathematical expression for general rebate warranty is given below

$$c_a^*(x) = \begin{cases} c_a & x < w_1 \\ c_a \frac{w_2 - x}{w_2 - w_1} & w_1 \le x \le w_2 \\ 0 & x > w_2. \end{cases} \tag{3.1}$$

The expression above states that if the failure occurs before w_1, the cost due to free replacement warranty is c_a. On the other hand if the failure occurs after time w_1 but before w_2 then the cost due to pro-rata warranty is in proportion to the difference between failure time and w_2, which is decreasing in nature. If the failure occurs after w_2 no cost due to warranty is being incurred. Since we use log lifetimes, therefore according to general rebate warranty policy the cost of accepting an item with log-lifetime t is

$$c_a^*(t) = \begin{cases} c_a & t < \ln w_1 \\ c_a \frac{w_2 - e^t}{w_2 - w_1} & \ln w_1 \leq t \leq \ln w_2 \\ 0 & t > \ln w_2. \end{cases} \tag{3.2}$$

Therefore the expected warranty cost per unit is given by

$$w(\theta) = c_a \left(\frac{w_2 F_T(\ln w_2) - w_1 F_T(\ln w_1)}{w_2 - w_1} - \frac{1}{w_2 - w_1} \int_{\ln w_1}^{\ln w_2} e^t f_T(t) dt \right). \tag{3.3}$$

Hence the expected warranty (acceptance) cost if n out of N items are put on test is given by

$$C_w = (N - n) w(\theta) \left(1 - \Phi\Big(\frac{\sigma(u_p + k)}{\sqrt{V}} \Big) \right). \tag{3.4}$$

From the literature, rejection cost usually is taken as cost due to units that are not tested (Hsieh and Lu, 2013). Thus if c_r is the cost per unit for the items that are not put on test, then the average cost of rejecting a lot is given by

$$C_r = (N - n) c_r \Phi\Big(\frac{\sigma(u_p + k)}{\sqrt{V}} \Big). \tag{3.5}$$

The expression for expected log-time of the test can be written as

$$E[\tau] = \int_0^\infty t f_{l:n-1}(t;\theta) dt + \int_0^{T_0} \big(F_{l:n-1}(t;\theta) - F_{r:n-1}(t;\theta) \big) dt; \tag{3.6}$$

where, $F_{r:n-1}(t;\theta) = \sum_{i=r}^{n-1} \binom{n-1}{i} F_T(T_0)^i \big(1 - F_T(T_0)\big)^{n-i-1}$.

Thus if c_t be the cost per unit, the expression for expected time consumption cost is given by $C_t = c_t E[\tau]$. Also if c_i is the unit cost of inspection, the average cost of inspection can be written as $C_i = n c_i$. The aggregate cost function is

$$\begin{aligned} TC(n, r, l, T_0) &= C_w + C_r + C_t + C_i \\ &= (N - n) w(\theta) \left(1 - \Phi\Big(\frac{\sigma(u_p + k)}{\sqrt{V}} \Big) \right) + (N - n) c_r \Phi\Big(\frac{\sigma(u_p + k)}{\sqrt{V}} \Big) + c_t E[\tau] + n c_i \\ &= (N - n) \left(w(\theta) + \big(c_r - w(\theta) \big) \Phi\Big(\frac{\sigma(u_p + k)}{\sqrt{V}} \Big) \right) + c_t E[\tau] + n c_i \end{aligned}$$

Therefore, the optimal design problem can be expressed as follows:

$$\textit{minimize } TC(n, r, l, T_0)$$

$$\textit{subject to } \frac{V}{\sigma^2} \left(\frac{z_\alpha - z_{1-\beta}}{u_{p_\alpha} - u_{p_\beta}} \right)^2 - 1 = 0.$$

The equality constraint as also shown in (2.8) ensures that the already agreed upon values pertaining to producer's and consumer's risks are being maintained.

4 Determining the optimal solution

The nature of the optimization problem mentioned in Section 3 for determining the optimal life testing plan is fairly complex. The problem being both non-linear and mixed-integer enhances the complexity of the problem. Due to the complex nature of the mathematical functions involved in framing various costs that constitute the objective function, integer inputs for the values of n, r and l are required. Hence to reduce the complexity of the problem, instead of using n as a decision variable we use $p_n = \frac{n}{N}$. To retain the integer nature of n, we replace n with $\lfloor p_n N \rfloor$, where $\lfloor . \rfloor$ represents greatest integer or the floor function. Similarly, instead of r as a decision variable we use the degree of censoring $q = 1 - \frac{r}{n}$ and instead of l use $q_1 = 1 - \frac{l}{r}$ and replace r and l with $\lfloor (1-q)n \rfloor$ and $\lfloor (1-q_1)r \rfloor$ respectively to retain their discrete nature. The continuous nature of p_n ($p_n \in [0,1]$), q ($q \in [0,1]$) and q_1 ($q_1 \in [0,1]$) transforms the problem to a nonlinear programming problem where the traditional algorithms such as augmented Lagrangian can be used to find the optimal solution. Therefore the transformed problem can be written as follows

$$\textit{minimize } TC(p_n, q, q_1, T_0)$$

$$\textit{subject to } \frac{V}{\sigma^2}\left(\frac{z_\alpha - z_{1-\beta}}{u_{p_\alpha} - u_{p_\beta}}\right)^2 - 1 = 0.$$

In order to extract the optimal value of n, r and l from the solution obtained by solving the above problem we again take the help of the floor function. The procedure followed to solve the problem can be summarized using the following steps:

Algorithm 1: Finding the optimal design.

Input: p, α, β, p_α, p_β, N, w_1, w_2 and unit costs

Output: n^*, r^*, l^*, ${X_0}^*$, and TC^*

1 Define functions F_T, f_T and $\mathscr{L}(p)$

2 Fix (α, p_α) and (β, p_β) in $\mathscr{L}(p)$ to find k and and the constraint function

3 Define C_w, C_r, C_t and C_i and hence the objective function $TC(n, r, l, T_0)$

4 Consider $p_n = \frac{n}{N}$ as a decision variable and subsequently $q = 1 - \frac{r}{n}$ and $q_1 = 1 - \frac{l}{r}$

5 Replace n with $\lfloor p_n N \rfloor$, r with $\lfloor (1-q)n \rfloor$ and l with $\lfloor (1-q_1)r \rfloor$ transform the objective function from $TC(n, r, l, T_0)$ to $TC(p_n, q, q_1, T_0)$

6 Minimize the objective function with respect to the given constraint to find the optimal values of (${p_n}^*$, q^*, ${q_1}^*$, ${T_0}^*$, TC^*) using non-linear optimization algorithms such as augmented Lagrangian

7 Obtain $n^* = \lfloor {p_n}^* N \rfloor$, $r^* = \lfloor (1-q)n^* \rfloor$, $l^* = \lfloor (1-q_1)r^* \rfloor$ and ${X_0}^* = e^{{T_0}^*}$ to find the optimal design $(n^*, r^*, l^*, {X_0}^*)$

The producer and the consumer through a joint agreement decides on the values of p_α and p_β. But for the purpose of our study we used the values from MIL-STD-105D (U D of Defense, 1963) which is a common practice followed in the literature (Schneider, 1989; Balasooriya and Low, 2004; Bhattacharya *et al*, 2015). Five different choices of p_α and p_β are used for the study. α and β values are also kept at 0.05 and 0.1 respectively since these are the most favored choices of α and β found in the literature. For computational purpose we have used the values of the parameters as $\mu = 5.2116$ and $\sigma = 0.4289$. The values represent the estimated parameter values of a real life data set used in Section 6 of the paper. Since choice of parameters is very important for designing a life test plan, it seems appropriate to use real life values for the parameters. The expression for total cost is determined considering the following unit costs $c_a = 0.15$ (unit cost of acceptance), $c_r = 0.80$ (unit cost of rejection), $c_t = 0.08$ (unit cost of time consumption) and $c_i = 0.05$ (unit cost of inspection). The *auglag* function from *nloptr* package in *R 3.2.2* was used to solve the problem.

The *nloptr* package in R addresses non-linear optimization problems with equality or inequality constraints which can be linear or non-linear in nature. The *auglag* within *nloptr* function uses augmented Lagrangian minimization algorithm for optimizing nonlinear objective functions with constraints. This method modifies the given objective function by combining the constraint function to it. The modified objective function is then fed to another optimization algorithm. Results are calculated for each pair of p_α and p_β values. The solutions thus obtained are stated in Table 1 below. In the above illustration the lot size (N) is assumed to be 500 and the warranty period is also kept fixed. The effect in optimal design due change in lot size and the effect due to change in warranty period is observed in Section 6.

Table 1: Type-I generalized hybrid censored life testing plans for given values of α, β, p_α, and p_β.

(α, β)					(0.05, 0.1)					
(p_α, p_β)	k	p_n^*	n^*	q^*	r^*	${q_1}^*$	l^*	${T_0}^*$	${X_0}^*$	TC^*
(0.02090, 0.07420)	3.130	0.1282	64	0.1225	56	0.1264	49	3.0211	20.5146	28.0096
(0.0190, 0.05350)	3.607	0.2217	110	0.1763	91	0.0945	82	2.3033	10.0069	27.9443
(0.00284, 0.03110)	4.509	0.0679	33	0.1172	29	0.1176	26	3.0616	21.3616	28.2577
(0.00654, 0.04260)	3.963	0.0802	40	0.1520	34	0.1316	29	2.6059	13.5438	28.4667
(0.03190, 0.09420)	2.802	0.1464	73	0.1507	62	0.1486	52	2.0639	7.8769	28.0501

The results obtained in Table 1 can be explained using the following example: Consider a lot of size (N) 500. Before starting the process of finding the optimal sampling plan, (α, β) values are fixed at (0.05,0.1) and the corresponding (p_α, p_β) values are fixed at (0.02090, 0.07420). After using Algorithm 1 to the design problem, the following optimal solution is obtained (${T_0}^* = 3.0211$, ${p_n}^* = 0.1282$, $q^* = 0.1225$, ${q_1}^* = 0.1264$) which after transformation translates to ($X_0^* = 20.5146$, $n^* = 64$, $r^* = 56$, $l^* = 49$). So under the given setup, for life testing of a lot, 64 (n^*) items are to be put on test simultaneously. The test is required to be terminated either when 20.5146 (X_0^*) units of time has elapsed or when 56 (r^*) failures are observed. But the test can be terminated after 20.5146 time units only when a minimum of 49 (l^*) items on test has failed. In case 20.5146 units of time elapse before 49 failures are observed, the test runs till 49^{th} failure is recorded.

As mentioned earlier, Type-I hybrid censoring is a special case of generalized Type-I hybrid censoring. When $l^* = 0$, generalized Type-I hybrid censoring becomes Type-I hybrid censoring. In case of Type-I hybrid censoring, the experiment gets terminated either when time (${X_0}^*$) is reached or when (${r^*}^{th}$) failure is observed. The concept of observing minimum l^* failures before terminating the experiment is not considered in this case which is one of the limitations of Type-I hybrid censoring as mentioned in Section 1. Considering $l^* = 0$ in the aforementioned setup, the results for Type-I hybrid censoring is obtained. The results thus obtained are recorded in Table 2. It is interesting to observe that when the condition of terminating an experiment after observing at least l^* failures is relaxed, the values of ${X_0}^*$, n^* and r^* increases.

Table 2: Type-I hybrid censored life testing plans when $l^* = 0$.

(α, β)				(0.05, 0.1)				
(p_α, p_β)	k	p_n^*	n^*	q^*	r^*	${T_0}^*$	${X_0}^*$	TC^*
(0.02090, 0.07420)	3.130	0.2725	136	0.1052	121	3.7397	42.0877	27.6796
(0.0190, 0.05350)	3.607	0.3300	165	0.1142	146	4.0414	56.9085	27.4661
(0.00284, 0.03110)	4.509	0.1468	73	0.0974	66	3.8428	46.6585	28.1433
(0.00654, 0.04260)	3.963	0.1643	82	0.1137	72	3.9088	49.8396	28.0771
(0.03190, 0.09420)	2.802	0.3384	169	0.1327	146	3.7794	43.7930	27.4366

In Table 3 CPU times (in seconds) for both Type-I generalized hybrid censored life testing plans and Type-I hybrid censored life testing plans when $l^* = 0$ is put forth. The CPU times are calculated using *proc.time* function in *R*. *proc.time* returns user, system, and elapsed times for a process. The user time is the time taken by CPU to execute the user instructions. The system time is the time taken by CPU to process the instructions. User and system time together represent the CPU time involved to complete the process on the other hand elapsed time is the total clock time required to complete the process. One observation that can be made from Table 3 is that the user, system, and elapsed times for determining Type-I hybrid censored life testing plans (HCLTP) is slightly higher than Type-I generalized hybrid censored life testing plans (GHCLTP).

Table 3: Computational times for Type I GHCLTP and Type I HCLTP.

(α, β)=(0.05, 0.1)	Type I GHCLTP			Type I HCLTP		
(p_α, p_β)	User time	System time	Elapsed time	User time	System time	Elapsed time
(0.02090, 0.07420)	1418.46	1.67	1467.32	1465.79	4.49	1565.88
(0.0190, 0.05350)	2087.89	1.65	2127.62	2183.75	5.62	2292.95
(0.00284, 0.03110)	1036.73	2.79	1111.23	1175.49	3.64	1252.14
(0.00654, 0.04260)	1064.36	2.1	1103.24	1265.33	4.58	1380.56
(0.03190, 0.09420)	1474.55	2.12	1642.93	1663.13	5.5	1718.17

5 Monte Carlo simulation

In order to validate the model, a rigorous simulation study is conducted by considering producer's risk. This becomes important since the life testing plans are developed using distribution of $\hat{\mu} - k\hat{\sigma}$ statistic which is asymptotically normal. Hence to validate whether the model holds true for finite sample sizes, Monte Carlo simulation is conducted for plans computed in Table 1. In order to compute the optimum design in the above section $L^{'} = {F_T}^{-1}(p_\alpha)$ is considered as acceptance criterion. For each solution set $(n^*, r^*, l^*, {T_0}^*)$, $10,000$ data sets are generated keeping α, β, p_α, p_β fixed. The maximum likelihood estimates are obtained using equation (2.4) for each of the data sets. Now using the lot acceptance criterion $\hat{\mu} - k\hat{\sigma} > L^{'}$ for each data set to reject the lots, the proportion of rejection should come close to α. The results can be seen in Table 4 below. The method followed for simulation study is elaborated in Algorithm 2. The results validate that the model works reasonably well even when finite sample sizes are considered.

Table 4: Results of simulation study conducted for each set of solution in Table 1.

(α, β)				(0.05, 0.1)			
(p_α, p_β)	k	n^*	r^*	l^*	X_0^*	TC^*	$\hat{\alpha}$
(0.02090, 0.07420)	3.130	64	56	49	20.5146	28.0096	0.0471
(0.0190, 0.05350)	3.607	110	91	82	10.0069	27.9443	0.0509
(0.00284, 0.03110)	4.509	33	29	26	21.3616	28.2577	0.0537
(0.00654, 0.04260)	3.963	40	34	29	13.5438	28.4667	0.0469
(0.03190, 0.09420)	2.802	73	62	52	7.8769	28.0501	0.0517

Algorithm 2: Monte Carlo simulation.

Input: μ, σ, α, β, p_α, p_β, k, ${T_0}^*$, n^*, r^* and l^*

Output: $\hat{\alpha}$

Define ${F_T}^{-1}$

Initialization: coefficient=matrix(0,rows=10,000, columns=2)

for $i \leftarrow 1$ **to** $10,000$ **by** 1 **do**

  for $j \leftarrow 1$ **to** n^* **by** 1 **do**

    $\pi_j \sim Uniform(0, 1)$

    $x_j = {F_T}^{-1}(\pi_i)$

  Find MLEs of the parameters ($\hat{\mu}$, $\hat{\sigma}$) using x_js

  coefficient[i,]=($\hat{\mu}$, $\hat{\sigma}$)

Initialization: count=0

for $i \leftarrow 1$ **to** $10,000$ **by** 1 **do**

  if *coefficient* $[i, 1]$-$k\times$*coefficient*$[i, 2] > {F_T}^{-1}(p_\alpha)$ **then**

    count=count+1

$\hat{\alpha} = \frac{10{,}000 - count}{10{,}000}$

6 Sensitivity analysis

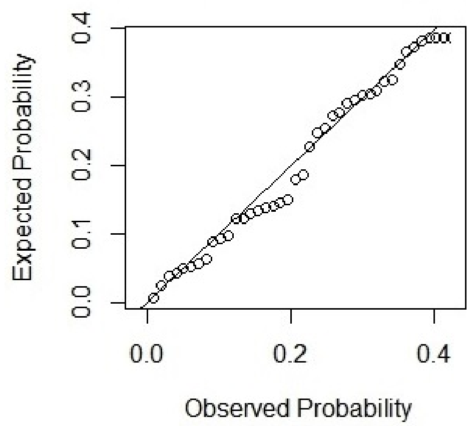
While going through the exercise of development of this life testing plan, the parameters pertaining to the extreme value distribution of T played a major role. Therefore, to investigate the effect of mis-specification of parameters in the optimum design and the total cost, a sensitivity analysis study is incorporated. For the purpose of this study a historical real life data set from Lawless (2003) is used.

Each point of the data set represents the number of thousand miles at which different locomotive controls have failed in a life testing experiment. The data set involves a sample size (n) of 96 and the test was aborted when 135 thousand miles (X_0) elapsed and 37 failures were observed. So as we can observe that number of traversed miles is taken as a proxy for failure times for the life testing experiment. The data depicting the failure times (in thousand of miles) of the units failed during the life test experiment is represented as follows:

22.5, 37.5, 46.0, 48.5, 51.5, 53.0, 54.5, 57.5, 66.5, 68.0, 69.5, 76.5, 77.0, 78.5, 80.0, 81.5, 82.0, 83.0, 84.0, 91.5, 93.5, 102.5, 107.0, 108.5, 112.5, 113.5, 116.0, 117.0, 118.5, 119.0, 120.0, 122.5, 123.0, 127.5, 131.0, 132.5 and 134.0.

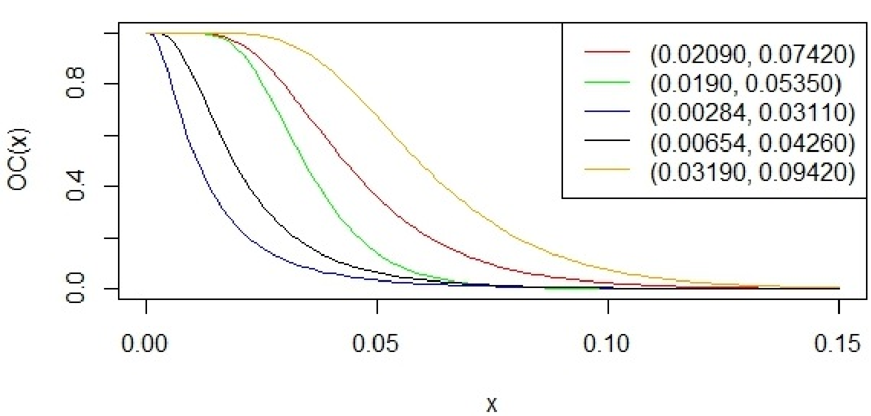
Given the data set, the parameter values $(\hat{\mu}, \hat{\sigma})$ are estimated along with their corresponding standard errors assuming the lifetime to follow Weibull distribution. The estimates (corresponding standard errors) of μ and σ are found to be 5.2116(0.0898) and 0.4289(0.0664) respectively. In order to ensure that the distributional assumption (to arrive at the estimates) holds true, PP plot is drawn. The PP plot depicted in Figure 5 shows a good fit and hence validates that the distributional assumption holds true.

Figure 5: PP plot of Lawless (2003) data set using Weibull distribution.



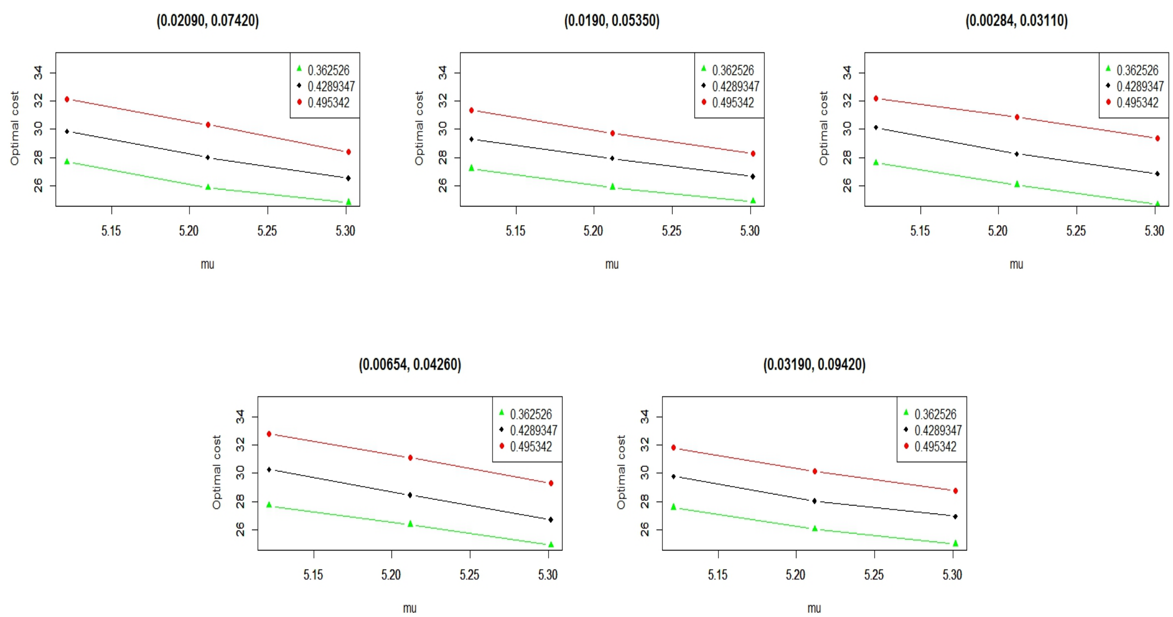
For the purpose of sensitivity analysis, the estimates and the standard errors are used to arrive at three sets of values (estimates $\pm$ standard error) for each parameter. Using $\alpha = 0.05$, $\beta = 0.1$ and five pairs of (p_α, p_β) values from MIL-STD-105D (U D of Defense, 1963) the optimal design corresponding to each of the nine set of parameters for each pair of (p_α, p_β) is determined. The five pairs of (p_α, p_β) values result in five different OC curves depicted in Figure 6.

Figure 6: OC curves for different choices of (p_α, p_β) values.



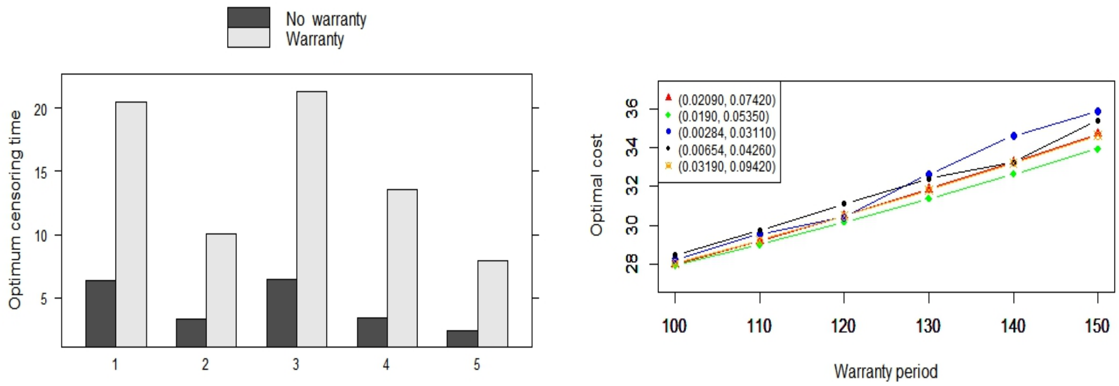
From Table 5 in Appendix it can be observed that the optimal design changes with change in parameter values although no pattern or trend in the change is visible. The result reinstates the importance of the parameter values for optimal design plan. But on the other hand, a clear trend emerges from the values of optimum cost which can be seen from Figure 7. As parameter μ increases keeping the value of σ fixed, the optimum cost decreases which is evident from the downward sloping lines in the figure for every combination of σ and (p_α, p_β) values. Again for each (p_α, p_β) it can be witnessed that the lines shift upwards parallely with increase in the value of σ which signifies that the optimal cost increases with increase in σ if μ is kept fixed. From the above results one can infer that maintaining higher average lifetime of a product with minimum variance can help in reducing the cost incurred which includes warranty cost. (p_α, p_β) values are mentioned at the top of each graph in Figure 7 and σ values (representing their respective colors) are mentioned in the legend.

Figure 7: Change in optimal cost due to change in parameters.



Since the transformation $p_n = \frac{n}{N}$ is made while solving the optimal design problem, the impact in the optimal design due to change in lot size (N) is also analyzed. The results depicted in Table 6 in Appendix summarizes that optimal design is not significantly impacted by the lot size although it is obvious for the optimal cost to increase with the increase in lot size. Furthermore, one of the important aspects of the cost function chosen for the analysis is the warranty cost, hence change in optimal design due to inclusion of warranty cost is looked at. In Figure 8 it can be witnessed that the inclusion of warranty cost increases the optimal censoring time while other variables (n, r, l) remain fairly consistent optimally. It can also be seen that the change in the duration of warranty impacts the optimal total cost. The optimal cost increases with increase in duration of warranty in almost a linear manner.

Figure 8: Change in optimal design due to warranty (1,2,...,5 represents (p_α, p_β) values in the same order as it appears in Table 1).



8 Conclusion

In this work, a method is proposed to arrive at an optimum life testing plan under Type-I generalized hybrid censoring. Weibull lifetime model is considered in the context of this study, however under the ambit of the developed methodology other lifetime distributions of log-location scale family can also be used. The work tries to formulate optimum reliability acceptance sampling plans from a management perspective which makes it valuable in dealing with real life problems pertaining to product quality management. The proposed approach will help in deciding on the acceptance of a batch or lot keeping the cost under consideration which also includes warranty cost to be incurred. A cue from this model may also provide some input in deciding the warranty policy for the future. An extensive simulation study is conducted to validate the model proposed and the results obtained are as desired. To get further insights from the model a rigorous sensitivity study is conducted. The results from the study highlights the importance of the parameters and accesses the behavior of the optimal cost due to parameter changes. As warranty cost is included in the objective function to develop a meaningful model for consumer durable products, the impact of its inclusion in the model is looked at. Insights on the behavior of optimal cost due to change in period of warranty is also highlighted.

Warranty claims lead to rework, as a result of which cost in terms of efforts, time and money has to be borne by the company. Hence for consumer durable products it is important to ensure that the cost due to warranty is induced in designing the acceptance sampling plan. Therefore, from quality management perspective this study takes a small step forward in the direction of addressing a practical problem. As a scope for future research, the proposed method can also be studied under different censoring schemes with appropriate lifetime distributions. Many a times it may be realistic to assume that the parameters involved arise out of some prior distribution because of uncertainty engaged in the parameter values. Therefore, future research can also study the problem under a Bayesian setup.

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Appendix

Table 5: Results of sensitivity analysis using Lawless (2003) failure data of locomotive controls.

(p_α, p_β)	$(\widehat{\mu}, \widehat{\sigma})$	${p_n}^*$	n^*	q^*	r^*	${q_1}^*$	l^*	T_0^*	X_0^*	TC^*
	(5.12173, 0.36253)	0.1286	64	0.1023	57	0.1025	51	3.0866	21.9034	27.6757
	(5.21162, 0.36253)	0.1335	66	0.1311	58	0.1334	50	3.3431	28.3069	25.8861
	(5.30151, 0.36253)	0.1357	67	0.1710	56	0.1891	45	3.1384	23.0668	24.8308
	(5.12173, 0.42893)	0.1267	63	0.1240	55	0.1247	48	2.8143	16.6811	29.8627
(0.02090, 0.07420)	(5.21162, 0.42893)	0.1281	64	0.1225	56	0.1269	49	3.0211	20.5146	28.0096
	(5.30151, 0.42893)	0.1288	64	0.3158	44	0.2401	33	3.4389	31.1529	26.5535
	(5.12173, 0.49534)	0.1278	63	0.2463	48	0.1281	42	2.9979	20.0453	32.1274
	(5.21162, 0.49534)	0.1270	63	0.2874	45	0.2254	35	2.7085	15.0073	30.3198
	(5.30151, 0.49534)	0.1288	64	0.1648	53	0.1547	45	3.0466	21.0451	28.4073
	(5.12173, 0.36253)	0.2262	113	0.0884	103	0.0884	93	2.3246	10.2230	27.2180
	(5.21162, 0.36253)	0.2349	117	0.0998	105	0.0999	95	2.2936	9.9109	25.8978
	(5.30151, 0.36253)	0.2445	122	0.1346	105	0.1197	93	2.3138	10.1137	24.9106
	(5.12173, 0.42893)	0.2171	108	0.1261	94	0.0885	86	2.2929	9.9039	29.3028
(0.0190, 0.05350)	(5.21162, 0.42893)	0.2217	110	0.1763	91	0.0944	82	2.3032	10.0068	27.9442
	(5.30151, 0.42893)	0.2267	113	0.1772	93	0.1582	78	2.0122	7.4798	26.6654
	(5.12173, 0.49534)	0.2139	106	0.1394	92	0.1254	80	2.3860	10.8701	31.3623
	(5.21162, 0.49534)	0.2141	107	0.1318	92	0.1263	81	2.1232	8.3582	29.7227
	(5.30151, 0.49534)	0.2185	109	0.1723	90	0.1664	75	2.9473	19.0555	28.3160
	(5.12173, 0.36253)	0.0574	28	0.0838	26	0.0929	23	3.0075	20.2364	27.6111
	(5.21162, 0.36253)	0.0615	30	0.0997	27	0.1068	24	2.9180	18.5053	26.0701
	(5.30151, 0.36253)	0.0577	28	0.1830	23	0.1536	19	2.9869	19.8248	24.6750
	(5.12173, 0.42893)	0.0583	29	0.1423	25	0.1094	22	2.6372	13.9752	30.1249
(0.00284, 0.03110)	(5.21162, 0.42893)	0.0678	33	0.1172	29	0.1176	26	3.0615	21.3616	28.2577
	(5.30151, 0.42893)	0.0696	34	0.1927	28	0.1106	24	3.2513	25.8245	26.8612
	(5.12173, 0.49534)	0.0651	32	0.1167	28	0.1267	25	2.9235	18.6074	32.1644
	(5.21162, 0.49534)	0.0635	31	0.1849	25	0.1885	21	2.7514	15.6644	30.8612
	(5.30151, 0.49534)	0.0705	35	0.1924	28	0.1262	24	2.5093	12.2974	29.3871
	(5.12173, 0.36253)	0.0810	40	0.1643	33	0.1165	29	1.9908	7.3219	27.7129
	(5.21162, 0.36253)	0.0807	40	0.2418	30	0.1337	26	2.0291	7.6069	26.3831
	(5.30151, 0.36253)	0.0825	41	0.1794	33	0.1302	29	2.0155	7.5048	24.9165
	(5.12173, 0.42893)	0.0813	40	0.2210	31	0.1726	26	1.9462	7.0025	30.2629
(0.00654, 0.04260)	(5.21162, 0.42893)	0.0802	40	0.1520	34	0.1314	29	2.6059	13.5438	28.4666
	(5.30151, 0.42893)	0.0832	41	0.1632	34	0.1754	28	2.1069	8.2234	26.7260
	(5.12173, 0.49534)	0.0825	41	0.1485	35	0.1187	30	1.9453	6.9959	32.7969
	(5.21162, 0.49534)	0.0503	25	0.1541	21	0.1478	18	2.4238	11.2895	31.0905
	(5.30151, 0.49534)	0.0901	45	0.1352	38	0.1695	32	2.6110	13.6130	29.3001
	(5.12173, 0.36253)	0.1461	73	0.1194	64	0.1023	57	2.0963	8.1362	27.5712
	(5.21162, 0.36253)	0.1518	75	0.1182	66	0.1042	59	2.0608	7.8529	26.0635
	(5.30151, 0.36253)	0.1527	76	0.1472	65	0.1513	55	2.0984	8.1537	25.0104
	(5.12173, 0.42893)	0.1458	72	0.1205	64	0.1220	56	2.0094	7.4591	29.7772
(0.03190, 0.09420)	(5.21162, 0.42893)	0.1464	73	0.1507	62	0.1486	52	2.0639	7.8769	28.0501
	(5.30151, 0.42893)	0.1470	73	0.1671	61	0.1656	51	2.3076	10.0499	26.9617
	(5.12173, 0.49534)	0.1446	72	0.1897	58	0.1574	49	2.0075	7.4447	31.7905
	(5.21162, 0.49534)	0.1442	72	0.1437	61	0.1641	51	2.2112	9.1263	30.1562
	(5.30151, 0.49534)	0.1443	72	0.2484	54	0.2380	41	2.0747	7.9627	28.7912

Table 6: Optimal design for different lot sizes.

(p_α, p_β)	N	${p_n}^*$	n^*	q^*	r^*	${q_1}^*$	l^*	T_0^*	X_0^*	TC^*
	200	0.3243	64	0.2185	50	0.1103	45	2.9875	19.8363	10.9387
	300	0.2136	64	0.1306	55	0.1181	49	2.9990	20.0664	16.6291
(0.02090, 0.07420)	400	0.1604	64	0.1485	54	0.0910	49	2.9561	19.2229	22.3193
	500	0.1281	64	0.1224	56	0.1264	49	3.0211	20.5146	28.0096
	600	0.1048	62	0.1003	56	0.1035	50	3.0234	20.5616	33.9523
	200	0.5524	110	0.1473	94	0.1188	83	2.3103	10.0782	10.6794
	300	0.3676	110	0.1302	95	0.1286	83	2.4070	11.1012	16.4343
(0.0190, 0.05350)	400	0.2763	110	0.1299	96	0.1301	83	2.3491	10.4759	22.1893
	500	0.2217	110	0.1762	91	0.0944	82	2.3032	10.0068	27.9442
	600	0.1834	110	0.1290	95	0.1294	83	2.3929	10.9459	33.6992
	200	0.1693	33	0.1039	30	0.1180	26	3.0548	21.2177	11.1649
	300	0.1102	33	0.1162	29	0.1210	25	3.0019	20.1244	16.8625
(0.00284, 0.03110)	400	0.0846	33	0.1120	30	0.1057	26	3.0428	20.9652	22.5601
	500	0.0678	33	0.1172	29	0.1176	26	3.0615	21.3616	28.2577
	600	0.0558	33	0.1041	30	0.1037	26	3.0498	21.1114	33.9553
	200	0.2125	42	0.2057	33	0.2136	26	2.6038	13.5154	11.1606
	300	0.1363	40	0.2631	30	0.2150	23	2.6149	13.6661	16.9594
(0.00654, 0.04260)	400	0.1004	40	0.2701	29	0.2133	23	2.6912	14.7507	22.7130
	500	0.0802	40	0.1520	34	0.1314	29	2.6059	13.5438	28.4666
	600	0.0681	40	0.2588	30	0.2171	23	2.6401	14.0148	34.2203
	200	0.3669	73	0.1067	65	0.2753	47	1.8676	6.4731	10.9072
	300	0.2439	73	0.1372	63	0.2090	49	2.0961	8.1344	16.6214
(0.03190, 0.09420)	400	0.1845	73	0.1248	64	0.2071	51	1.8873	6.6016	22.3357
	500	0.1464	73	0.1507	62	0.1486	52	2.0639	7.8769	28.0501
	600	0.1229	73	0.1705	61	0.1791	50	1.8221	6.1847	33.7643

Research Office

Indian Institute of Management Kozhikode

IIMK Campus P. O.,

Kozhikode, Kerala, India,

PIN - 673 570

Phone: **+91-495-2809237/238**

Email: research@iimk.ac.in

Web: https://iimk.ac.in/faculty/publicationmenu.php